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Corrigendum

Corrigendum and addendum to my paper concerning Kummer extensions with few roots of unity ☆

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Abstract

This note presents corrections and additions to my paper (J. Number Theory 41 (1992) 322–358).

MSC: 12F05; 12F10; 12F99

Introduction

The following result is claimed to be true in [1]. (The numbering is taken from [1], as is any unexplained terminology.)

Corollary 5.13. *Let K be an arbitrary field and $p > 0$ a prime number, other than the characteristic of K. Let $k \in \mathbb{N}^*$, $a_1, \ldots, a_k \in K^*$, and let $\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}$ denote fixed pth roots. Then*

$$[K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) : K] = |\langle \widehat{\sqrt[p]{a_1}}, \ldots, \widehat{\sqrt[p]{a_k}} \rangle| = |\langle \widehat{\widehat{a}}_1, \ldots, \widehat{\widehat{a}}_k \rangle|,$$

where $\langle \widehat{\sqrt[p]{a_1}}, \ldots, \widehat{\sqrt[p]{a_k}} \rangle$ (resp. $\langle \widehat{\widehat{a}}_1, \ldots, \widehat{\widehat{a}}_k \rangle$) is the subgroup of $K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})^*/K^*$ (resp. K^*/K^{*p}) generated by the cosets $\widehat{\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}}$ in the quotient group $K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})^*/K^*$ (resp. by the cosets $\widehat{\widehat{a}}_1, \ldots, \widehat{\widehat{a}}_k$ in the quotient group K^*/K^{*p}).

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Moreover, if $a_i \notin K^{*p}$ for all i, $1 \leqslant i \leqslant k$, then

$$[K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) : K] = p^k$$

if and only if the following condition is satisfied:

$$i_1, \ldots, i_k \in \mathbb{N} \text{ and } a_1^{i_1} \cdot \ldots \cdot a_k^{i_k} \in K^p \Rightarrow p \mid i_s \quad \text{for all } s, \ 1 \leqslant s \leqslant k.$$

In this case,

$$K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) = K(\sqrt[p]{a_1} + \cdots + \sqrt[p]{a_k}).$$

Also, in the Abstract of [1] it is stated that "if K is an arbitrary field and n is a prime number other than Char(K), then any extension $K \subseteq K(x_1, \ldots, x_k)$, where $k \in \mathbb{N}^*$ and $x_1, \ldots, x_k \in \bar{K}^*$ are such that $x_i^n \in K$ for all i, $1 \leqslant i \leqslant k$, is a finite Kummer extension of exponent n with few or with many roots of unity".

The first part of Corollary 5.13, as well as the above statement of the abstract are wrong, as the following simple counterexample shows.

Take $p = 3$, $k = 2$, $K = \mathbb{Q}$, $a_1 = a_2 = 2$, and take as third roots of a_1 and a_2 the complex numbers $\sqrt[3]{2}$ and $\zeta_3 \sqrt[3]{2}$, respectively. Then, we have neither $\zeta_3 \in \mathbb{Q}$ nor $\mu_3(\mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2})) \subseteq \{-1, 1\}$, i.e., the extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2})$ is neither a Kummer extension with few roots of unity, nor a Kummer extension with many roots of unity. Also, we have $[\mathbb{Q}(\sqrt[3]{2}, \zeta_3 \sqrt[3]{2}) : \mathbb{Q}] = 6$, but $|\langle \widehat{\sqrt[3]{2}}, \widehat{\zeta_3 \sqrt[3]{2}} \rangle| = 9$.

1. Corrigendum

The correct statement of Corollary 5.13 is the following one.

Corollary 5.13. *Let K be an arbitrary field, let $p > 0$ be a prime number other than the characteristic of K, let $k \in \mathbb{N}^*$, and let $a_1, \ldots, a_k \in K^*$. Then, either $\zeta_p \in K$, or there exist pth roots $\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k} \in \Omega$ such that $\mu_p(K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})) = \{1\}$. Moreover, if $[K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) : K] = p^k$, then the following statements hold:*

(1) $|\langle \widehat{\sqrt[p]{a_1}}, \ldots, \widehat{\sqrt[p]{a_k}} \rangle| = |\langle \widehat{\widehat{a}}_1, \ldots, \widehat{\widehat{a}}_k \rangle| = p^k$, *where $\widehat{\widehat{a}}$ denotes for any $a \in K^*$ its coset in the group K^*/K^{*p}.*
(2) *If $i_1, \ldots, i_k \in \mathbb{N}$ and $a_1^{i_1} \cdot \ldots \cdot a_k^{i_k} \in K^{*p}$, then $p \mid i_1, \ldots, p \mid i_k$.*
(3) $K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) = K(\sqrt[p]{a_1} + \cdots + \sqrt[p]{a_k})$.

Proof. By Examples 1.4(iii) and 3.1(ii), we have either $\zeta_p \in K$, or there exist specific pth roots $\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k} \in \Omega$ such that $\mu_p(K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})) = \{1\}$.

(1) Follows from Theorem 5.1.

(2) Clearly, the condition $[K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) : K] = p^k$ implies that $[K(\sqrt[p]{a_i}) : K] = p$ for all $i = 1, \ldots, k$. Now apply Corollary 5.2.

(3) Follows from Theorem 5.9. $\square$

2. Addendum

In this section, we give negative answers to two problems raised in [1], and present a result from [2] related to Corollary 5.13.

The next example shows that the answer to [1, Problem 5.11] is *no*.

Example 2.1. [4, Example 4.9] We have $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}, \sqrt[6]{72}) = \mathbb{Q}(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{72})$ [4, Corollary 4.7], but

$$[\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}, \sqrt[6]{72}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt[3]{3}) : \mathbb{Q}] = 6$$

$$< [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]\,[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}]\,[\mathbb{Q}(\sqrt[6]{72}) : \mathbb{Q}].$$

The next example shows that the answer to [1, Problem, p. 340] is *no*.

Example 2.2. Let $K = \mathbb{Q}$, $n = 4$, $k = 2$, $a_1 = 2^2 \cdot 3$, and $a_2 = 3 \cdot 5^2$. Since $[\mathbb{Q}(\sqrt[4]{a_1}^2) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{a_2}^2) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, we have $s_1 = s_2 = 2$. If we take $L = \mathbb{Q}(\sqrt[4]{a_1 a_2}) = \mathbb{Q}(\sqrt{2 \cdot 3 \cdot 5})$ or $L = \mathbb{Q}(i, \sqrt[4]{a_1 a_2})$, then

$$[K(\sqrt[4]{a_1}, \sqrt[4]{a_2}) : K(\sqrt[4]{a_1}^{s_1}, \sqrt[4]{a_2}^{s_2})] \neq [L(\sqrt[4]{a_1}, \sqrt[4]{a_2}) : L(\sqrt[4]{a_1}^{s_1}, \sqrt[4]{a_2}^{s_2})].$$

Remark 2.3. By [2, Theorem 1.4], Corollary 5.13 as stated in the previous section holds for any choice $\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k} \in \Omega$ of pth roots of unity, and not only for the specific ones such that $\mu_p(K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})) = \{1\}$; in other words, Corollary 5.13 in its original form [1] remains valid if the additional condition "$[K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k}) : K] = p^k$" is added. Moreover, in this case, the extension $K \subseteq K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})$ is $K^* \langle \sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k} \rangle$-Cogalois (see [3] for the concept of G-Cogalois extension), and consequently, the map $H \mapsto K(H)$ yields an isomorphism between the lattice of all subgroups containing K^* of the group $K^* \langle \sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k} \rangle$ and the lattice of all intermediate fields of the extension $K \subseteq K(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_k})$.

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